

Loop conditions

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Abstract

We discuss such Maltsev conditions that consist of just one linear equation, we call them loop conditions. To every such condition can be assigned a graph. We provide a classification of conditions with undirected graphs. It follows that the Siggers term is the weakest non-trivial loop condition.

1 Introduction

In 2010, M. Siggers [3] observed that any finite Taylor algebra has a Siggers term s satisfying $s(x, y, y, z, z, x) = s(y, x, z, y, x, z)$. Later, A. Kazda noticed that this is not true in general – free idempotent algebra modulo WNU3 $w(x, y, y) = w(y, x, y) = w(y, y, x)$ does not satisfy any non-trivial linear Maltsev condition consisted of one equation. Despite this, there is another strong Maltsev condition satisfied by any idempotent algebra: $t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y)$ [2].

Of course, this condition cannot consist of one equation, so there is a natural question – is the existence of a term satisfying a single non-trivial equation a strong Maltsev condition? We will find a whole class of such weakest conditions, the existence of a Siggers term is among them. Although we present the result under a theory of compatible relations, it is in fact a purely syntactical result – one could write the Siggers term by using any another term satisfying a linear non-trivial equation.

2 Loop conditions

Definition 1. *A loop condition is a requirement for an algebra or a variety for having a (at least unary) term t satisfying an equation of form*

$$t(\text{variables}) = t(\text{variables})$$

It is a specific type of strong Maltsev conditions, where can be more equations and they can involve term composition. An example of a loop conditions is the existence on a 6-ary Siggers term s satisfying

$$s(x, y, y, z, z, x) = s(y, x, z, y, x, z).$$

Loop conditions for algebras and for varieties are connected in the same way as usual Maltsev conditions are. If an algebra \mathbf{A} satisfy a loop condition then

the whole variety generated by \mathbf{A} satisfy it. Reversely, if a variety \mathcal{V} satisfy a loop condition, then the loop condition is satisfied by every algebra in \mathcal{V} , in particular a generator of \mathcal{V} . It follows that if one loop condition for an algebra implies another loop condition for an algebra the same implication holds for varieties and vice versa.

We assign an oriented graph $G_{\mathbb{C}}$ to a loop condition \mathbb{C} in the following way: Vertices are formed by the set of variables. Assume that the condition is of the form $t(u_1, \dots, u_n) = t(v_1, \dots, v_n)$ where symbols u_i, v_i are replaced by some variables. The edges of the graph are oriented pairs (u_i, v_i) without possible duplicity. If there are two oriented edges between two vertices in both directions, we consider it as an unoriented edge.

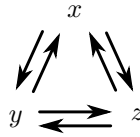


Figure 1: The graph of the Siggers term.

For example, the graph of the commutative term is just one unoriented edge. The graph of the Siggers term is an unoriented triangle.

Notice that a loop (v, v) in the graph assigned to a loop condition causes triviality – the condition can be satisfied by a projection then. Reversely, if the graph is loopless, the loop condition is non-trivial.

3 Graphs compatible with an algebra

Compatible relations are a standard tool for Maltsev conditions. In this case these relations are binary so we will oriented graphs will suffice and their visualisation is more suitable.

Definition 2. Consider an algebra \mathbf{A} with universe A . A k -ary relation R on A is formally a subset of A^n . By $R(x_1, \dots, x_k)$ we mean $(x_1, \dots, x_n) \in R$. In the case $k = 2$ we also see it as an (oriented) graph – A is the set of vertices and R is the set of edges.

A relation (graph) R is said to be *compatible* with \mathbf{A} if it is a subuniverse of the algebra \mathbf{A}^2 .

Following proposition explains the name of loop conditions and connects these conditions with compatible graphs.

Proposition 1. Let \mathcal{V} be a variety and \mathbb{C} be a loop condition. Following conditions are equivalent

- (i) \mathcal{V} satisfies \mathbb{C} .
- (ii) For every $\mathbf{A} \in \mathcal{V}$ and for every graph G compatible with A following holds:
If there is a graph homomorphism $G_{\mathbb{C}} \rightarrow G$ then G contain a loop.
- (iii) For every $\mathbf{A} \in \mathcal{V}$ and for every graph G compatible with A following holds:
If a subgraph of G is isomorphic to $G_{\mathbb{C}}$ then G contain a loop.

Proof. (i) \Rightarrow (ii) Let t be the term given by the loop condition. By compatibility, if we apply t to a tuple of edges, the result is also an edge. We will use the edges of G which are images of edges of $G_{\mathbb{C}}$ in the order from the construction of $G_{\mathbb{C}}$. By the loop condition the initial vertex is equal to the terminal vertex of the resulting edge so there is a loop in G .

(ii) \Rightarrow (iii) Trivial.

(iii) \Rightarrow (i) Let \mathbf{F} be the \mathcal{V} -free algebra generated by vertices of $G_{\mathbb{C}}$. Let E be the subuniverse of \mathbf{F}^2 generated by edges of $G_{\mathbb{C}}$. Set E forms edges of a graph compatible with F containing $G_{\mathbb{C}}$ as a subgraph. So by (iii) there is loop (a, a) in E . The pair (a, a) is generated by generators of E , so there is a term operation t taking edges of $G_{\mathbb{C}}$ as parameters and returning (a, a) . Thus in \mathbf{F} , the term t satisfies the loop condition if we plug generators of \mathbf{F} to it. So by universality of free algebra, the term t satisfies \mathbb{C} in general. \square

Corollary 1. *Assume loop conditions \mathbb{C}, \mathbb{D} . If there is a graph homomorphism $G_{\mathbb{C}} \rightarrow G_{\mathbb{D}}$, then \mathbb{C} implies \mathbb{D} . Namely \mathbb{C} implies \mathbb{D} if \mathbb{C} is a subgraph of \mathbb{D} .*

Finally, we recall a standard method for building compatible relations (graphs) from existing ones – pp-definitions and pp-powers. A relation R is *pp-definable* from relations R_1, \dots, R_n if it can be described by a formula using variables, existential quantifiers, conjunctions and predicates R_1, \dots, R_n . If R_1, \dots, R_n are compatible with an algebra, then also R is.

Using graph view, pp-definition of a k -ary relation R from graphs G_1, \dots, G_n sharing one set of vertices V is given by a graph on a set U of vertices with n types of edges and with k distinguished vertices u_1, \dots, u_k . Then vertices $v_1, \dots, v_k \in V$ are R -related if there is a mapping $U \rightarrow V$ which sends u_1, \dots, u_k to v_1, \dots, v_k respectively and which maps edges of i -th type to edges in G_i .

Now assume a pp-defined $k \cdot l$ -ary relation on A . We can understand it as a k -ary relation on the set A^l . In this case, we call it pp-power. If original relations are compatible with an algebra \mathbf{A} , then the pp-power is compatible with the algebraic power \mathbf{A}^l .

4 Unoriented case

In this section, we will focus on loop conditions with unoriented graph.

By proposition 1, loop equations with isomorphic graphs conditions are equivalent. Therefore, we will describe loop conditions by its graphs. Namely, we have loop conditions: edge (commutativity), triangle (Siggers), more generally n -clique and n -cycles.

At the begining, we can use just corollary 1 to get some implications between unoriented loop conditions. By definition, a loop condition cannot be empty, so every unoriented loop condition contains an unoriented edge. Thus edge is the strongest unoriented loop condition. Every bipartite graph can be homomorphically mapped to an edge, so all bipartite graphs loop conditions are equivalent.

Triangle implies 4-clique, it implies 5-clique and so on. Moreover every loopless graph is a subgraph of a clique large enough, so the chain of cliques reaches the weakest non-trivial loop conditions (not only non-oriented).

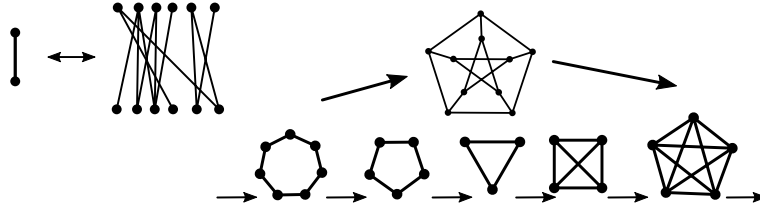


Figure 2: Scheme of trivial implications between unoriented loop conditions.

5-cycle can be mapped to triangle, 7-cycle can be mapped to 5 cycle and so on. Moreover every non-bipartite graph contains an odd cycle as a subgraph. Therefore the chain of odd cycles reaches the strongest unoriented non-bipartite loop conditions.

Our aim in the rest of the chapter is to reverse implications $(2n+3)$ -cycle to $(2n+1)$ -cycle and $n+2$ -clique to $n+3$ -clique, where $n \geq 1$. It will follow that all loopless unoriented non-bipartite graphs are equivalent as loop conditions and that they are weakest among all non-trivial loop conditions.

Proposition 2. *The $(2n+1)$ -cycle loop condition implies the $(2n+3)$ -cycle loop condition for any $n \geq 1$.*

Proof. Denote $k = 2n+1$. There is a graph homomorphism from the k^2 -cycle to the $(2n+3)$ -cycle because both cycles are odd and $(2n+1)^2 \geq 2n+3$. So it is sufficient to show that the k -cycle loop condition implies the k^2 -cycle one.

We use the proposition 1. \mathbf{A} is an algebra such that every compatible with \mathbf{A} containing a homomorphic image of k -cycle has a loop. We need to prove that every graph G containing a k^2 -cycle as a subgraph has a loop. Assume G fixed and the cycle consecutively formed by vertices v_0, v_1, \dots, v_{k^2} .

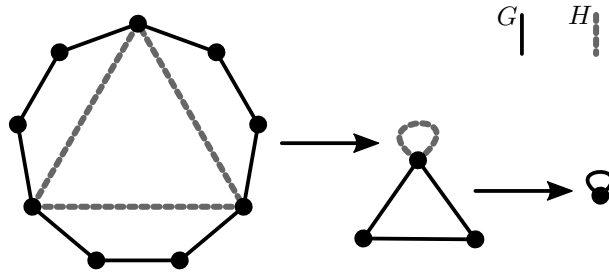


Figure 3: Getting a loop from 9-cycle using a triangle term condition.

We define another graph H : that there is an H -edge from x to y iff there is a G -walk from x to y composed of $2n+1$ edges. This is a pp-definition, so H is also compatible with \mathbf{A} . Vertices $v_0, v_k, v_{2k}, \dots, v_{k(k-1)}$ form a cycle in H of length k . By assumption, there is a loop in H . But a loop in H is a homomorphic image of a k -cycle in G . We use the assumption for the second time and get a loop in G . \square

Proposition 3. *The $(n+1)$ -clique loop condition implies the n -clique loop condition for any $n \geq 3$.*

Proof. As before, we use the proposition 1. Let \mathbf{A} be an algebra satisfying the $(n+1)$ -clique loop condition. Let G be a graph compatible with \mathbf{A} containing an n -clique a_1, a_2, \dots, a_n as a subgraph. It suffices to prove that G has to have a loop. We may suppose that G is symmetric because the symmetric part of G is pp-definable from G .

Let us pp-define a 4-ary relation R on A as follows. $R(u, v, x, y)$ if and only if there are elements $x_1, \dots, x_{n-2}, w \in A$ such that all vertices x_i are pairwise G -linked to each other, they are also G -linked with vertices x, y, v, w and moreover $G(u, w), G(w, x), G(v, y)$. From R we pp-define a relation (graph) $F(x, y) \Leftrightarrow \exists u \in A: R(u, u, x, y)$.

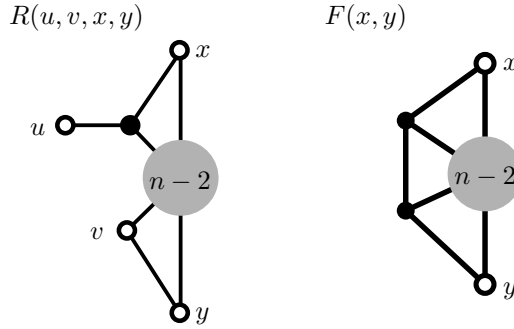


Figure 4: A vizualisation of the definitions of R and F .

Observe that the pp-definition of F is symmetric, co F itself is symmetric.

Claim 1. *If $i, j \in \{1, \dots, n\}$ and $i \neq j$ it holds $F(a_i, a_j)$.*

Proof. It holds $R(x_i, x_j, x_i, x_j)$ because there is an edge $G(x_i, x_j)$ and there remain clique of size $n-2$. \square

Claim 2. *Assume $u, v, x, y \in \{a_1, \dots, a_n\}$. To get $R(u, v, x, y)$, it is sufficient to meet one of the following conditions.*

- (a) $u \neq v$ and $x = y \neq v$.
- (b) $u = v$ and $x = u, y \neq x$.

To show it, assume variables x_1, \dots, x_{n-2}, w as in the definition of w .

- (a) We can set $w = v$ and variables x_i to different points a_i different from u, x .
- (b) We can set $w = y$ and variables x_i to different points a_i different from x, y .

The next claim immediatelly follows from the case (b) of the previous one.

Claim 3. *Assume $x, y \in \{a_1, \dots, a_n\}$. If $x \neq y$ then $F(x, y)$.*

Finally, we pp-define a relation (graph) Q on the algebra \mathbf{A}^2 as follows. $Q((u_1, u_2), (v_1, v_2))$ if and only if there are x_1, x_2, \dots, x_{n+1} such that for every pair of different indices i_1, i_2 with the exception of pairs $\{1, 2\}$ and $\{3, 4\}$ it holds $F(x_{i_1}, x_{i_2})$ and moreover $R(u_1, v_1, x_1, x_2)$ and $R(u_2, v_2, x_3, x_4)$.

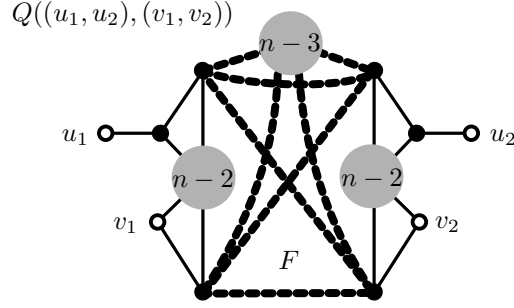


Figure 5: A vizualisation of the definitions of Q .

Claim 4. *If $u_1, u_2, v_1, v_2 \in \{a_1, \dots, a_{n+1}\}$ and $(u_1, u_1) \neg (v_2, v_2)$ then $Q((u_1, u_1), (v_2, v_2))$.*

To show it, assume variables x_1, \dots, x_{n+1} as in the definition of Q . We satisfy the condition just by using claims 2, 3. By F edges, all pairs of variables have to be differ with two possible exceptions $x_1 = x_2$ and $x_3 = x_4$. We analyse cases by equalites between u_1, u_2, v_1, v_2 . In each case we use the claim 2 for sufficient conditions for x_1, x_2, x_3, x_4 .

- (a) $u_1 \neq v_1$ and $u_2 \neq v_2$: We want to satisfy $x_1 = x_2 \neq v_1$ and $x_3 = x_4 \neq v_2$. At first we choose $x_1 = x_2$, then $x_3 = x_4$ different from v_2 and x_1 . It is possible since $n \geq 3$.
- (b) $u_1 = v_1$ and $u_2 \neq v_2$: We want to satisfy $x_1 = u_1$, $y_1 \neq x_1$, $x_3 = x_4 \neq v_2$. So we put $x_1 = u_1$, then choose a value for $x_3 = x_4$ different from x_1, v_2 and finally, we choose a value for x_2 different from x_1, x_3 .
- (c) $u_1 \neq v_1$ and $u_2 = v_2$ is analogous to the previous case.
- (d) $u_1 = v_1$ and $u_2 = v_2$ can not happen because $u_1, u_1) \neg (v_2, v_2)$.

In every case $u_1 = v_1$ or $u_2 = v_2$ so remaining variables can be completed.

Now, the graph Q is compatible with the algebra \mathbf{A}^2 since it is a pp-power. Q also contains a clique of size $n^2 \leq n + 1$. The n -clique loop condition holds also for \mathbf{A} , so there is a loop in Q .

The loop in Q means that there are elements x_1, \dots, x_{n+1} in A such that $F(x_i, x_j)$ whenever $i \neq j$. F is a compatible (pp-defined) graph, so there is a loop in F .

Finally, a loop in F causes an $(n + 1)$ -clique in G and afterwards the desired loop in G . \square

Corollary 2. *There are just three non-equivalent classes of loop conditions with unoriented graph G .*

- (1) G is bipartite,
- (2) G is non-bipartite and loopless,
- (3) G contains a loop. (trivial)

while (1) \Rightarrow (2) \Rightarrow (3) and (2) are weakest non-trivial loop conditions at all.

These implications cannot be reversed even for idempotent finite case. An example of an algebra satisfying (2) but not (1) is the algebra $(0, 1, 2, m)$ where $m(x, y, z) = x + y - z$.

5 Unoriented case – open

An oriented graph G is said to be *smooth* if every vertex has an incoming and an outgoing edge. G is said to have *the algebraic length 1* if there is no graph homomorphism from G to a directed cycle of a length greater than one. The following theorem holds for finite algebras [1].

Theorem 1. *Let \mathbf{A} be a finite algebra. Let G, H be weakly connected smooth graphs with algebraic length one. Then \mathbf{A} satisfies G loop condition if and only if it satisfies H loop conditions.*

Less formally, all connected smooth loop conditions with algebraic length 1 are equivalent for finite algebras. It gives a simpler weakest loop condition than triangle for finite algebras: $s(a, r, e, a) = s(r, a, r, e)$.

We have no idea whether the theorem or the corollary can hold in general. We recommend the following question for the beginning.



Figure 6: Open problems in loop conditions, known for finite algebras.

Open problem 1. *Let G be an unoriented triangle and H be a triangle with two unoriented edges and one oriented edge. Does G imply H ?*

References

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